SPECIAL RELATIVITY

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The popularity of the special theory of relativity stems from extraordinary predictions about time, distance, mass, Energy and the nature of space. What follows attempts to connect the concepts of time dilation with the equivalence of mass and energy and the concept of four dimensions. Relativity isn't a "fact", it is a set of ideas which can be used to interpret the results of experiments and observations and to make predictions in a consistent way, so far it has done so with great accuracy.

NOTE: Throughout this essay an assumption that NO ACCELERATION is going on is implicit, consideration of relativistic effects due to acceleration (or vice versa) are the realm of General relativity and Mach's Principle (which has nothing to do with the speed of sound).

There are two principles:

1. The speed of light, in empty space, is the same "c" independent of the motion of its source.

2. A principle best stated in several forms:

   Electrodynamic and mechanical effects do not have properties that require the concept of an absolute state of rest.

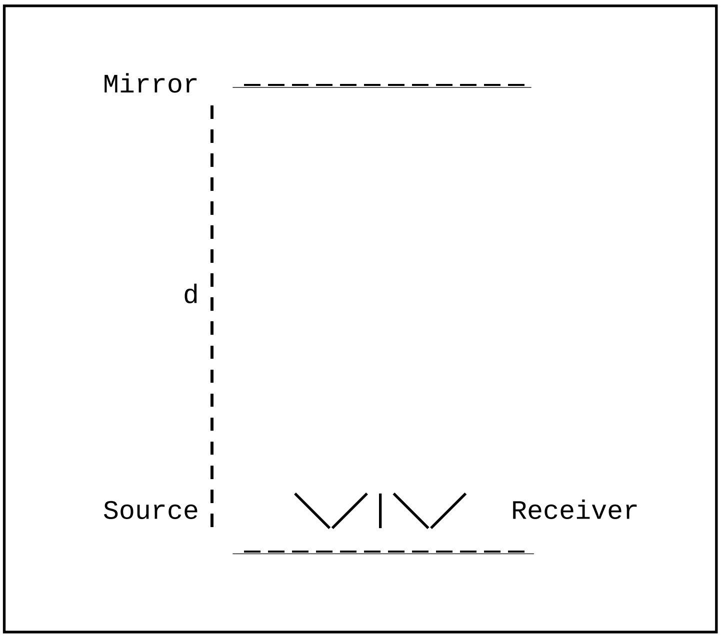
   The speed of an object cannot be determined IN ANY WAY without external reference. This is hard to prove since EVERY WAY of detecting motion should be checked and we don't know them all!

   No object is absolutely stationary (this is well known to the highway patrol).

   The concept of absolute (non-accelerating) motion is invalid.

PART 1 – Time and Distance Measurement

The apparatus I will use is a light source, coupled with a detector, both orientated to use a mirror " d" units away. There is also a stop watch attached to the apparatus that measures the time " t" taken for light to go from the source to the receiver.



Suppose this device moves past you at velocity "v" and orientated at " A" degrees to the direction of motion.

The following event is timed:

A ray leaves the source, is reflected by the mirror and is detected at the receiver.

The diagram shows the object at three stages in the event:

1. When the light is emitted from the source.
2. When it is reflected at the mirror.
3. When it is collected at the receiver.

   (see diagram above)

The heavy line indicates the path taken by the successful light rays collected at the receiver.

Let:

t_1 = the time to reach the mirror from the source
t_2 = the time to reach the receiver from the mirror
t = $t_1 + t_2$ the time for the whole transaction
c = the speed of the photons

The distance travelled to the mirror is ct_1 and from the mirror to the receiver is ct_2. The respective horizontal motions of the device are vt_1 and vt_2. Using the cosine rule:

$$c^2t1 = v^2t_1 + d^2 + 2vt_1d\cos(A) \qquad \text{(NOTE: } \cos(180 - A) = -\cos(A))$$

$$c^2t_2 = v^2t_2 + d^2 - 2vt_2d\cos(A)$$

rearranging the equation involving t_1 we have a quadratic equation which is solved to find t_1

$$t_1 = \frac{2vd\cos(A) + \sqrt{4d^2v^2\cos^2(A) + 4d^2(c^2 - v^2)}}{2(c^2 - v^2)}$$

$$t_1 = \frac{vd\cos(A) + d\sqrt{c^2 - v^2\sin^2(A)}}{(c^2 - v^2)}$$

Similarly

$$t_1 = \frac{-vd\cos(A) + d\sqrt{c^2 - v^2\sin^2(A)}}{(c^2 - v^2)}$$

Since the times taken are inherently positive we will use the positive instance for both t_1 and t_2, noting that

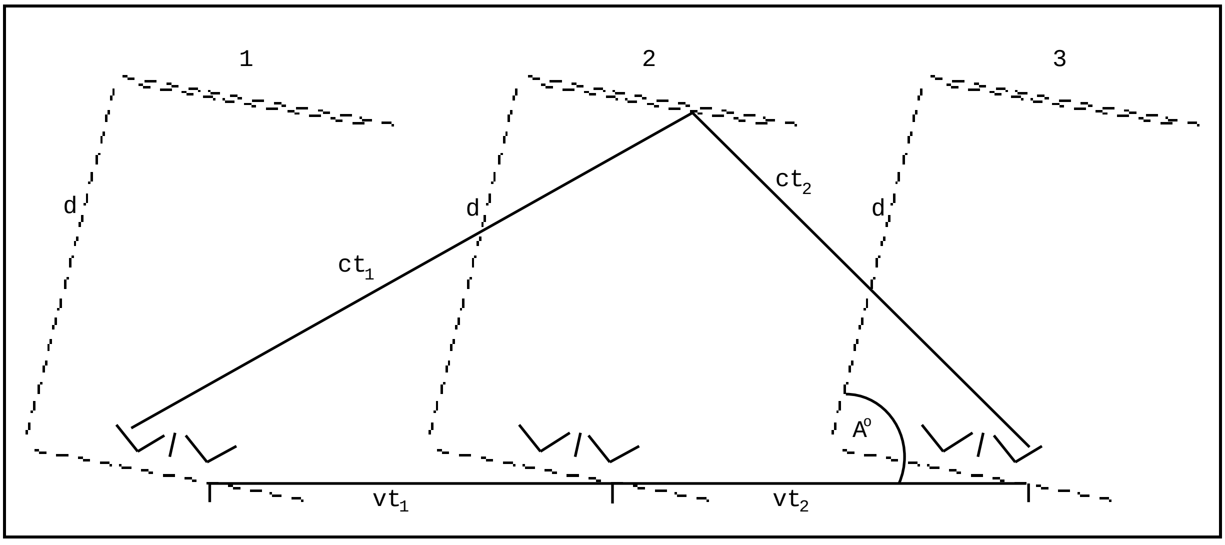
$$t = t_1 + t_2$$

the total time for the event is

$$t_1 = \frac{2d\sqrt{c^2 - v^2\sin^2(A)}}{(c^2 - v^2)}$$

This result indicates that the time taken for the event depends on both its speed and orientation.

Michelson and Morley proposed to detect the motion of the Earth through space by using a rotating configuration of two identical, perpendicular light paths with mirrors at the ends to reflect light from a common source to a receiver at which the interference of the rays could be observed. The time difference for the paths could be derived from the observations of changing interference patterns. No change in the interference patterns was detected. Hence if "t" is measured for my apparatus at various orientations there should be no variation either. Let us assume that this is the case.



Poincare suggested that not only the Michelson Morley experiment, but any experiment would be unable to determine the absolute velocity or direction of motion of an object, this is principle 2.

The second principle indicates that the time "t", recorded on the watch attached to the apparatus, for the event should be the same independent of the velocity of the apparatus, relative to a "stationary" observer, when the event was recorded. Note this does not mean that the value of "t", measured by a stationary observer with a stationary watch, is the same for any value of "v". In fact, the theory of relativity holds that this is not the case.

If we suppose that the value of "t", recorded on the watch attached to the apparatus, does not alter with the orientation or velocity of the device then there is an apparent contradiction with the algebra. Suppose that we let "d" change with the orientation and velocity of the apparatus so as to allow "t" to be constant. This may seem like a fairly desperate attempt to "fix" the algebra but there are common sense precedents for the idea. For example think of a string of marbles separated by springs, turn it into the flow of a current and the string will shorten. Matter is made up of atoms held by forces and perhaps it may be effected by some cosmic flow coming from a particular direction. Since all measuring sticks would be similarly effected you could not directly measure this alteration of length.

If "d" is to vary then so should "t" so that the speed of light remains constant as supposed in principle 1. What then of our measurements of "t" being constant? Recall that "t" is measured by a watch moving with the apparatus, if the passage of time is altered for the event then so will it be for the watch in such a way that no discrepancy will be detected. The value of "t" may however be different for an observer moving with respect to the apparatus. I will only suppose that the value of "t" depends on the relative velocity of the apparatus but not on its orientation.

To indicate that t depends on v we use t(v)
and that d depends on v and A we use d(v,A)

So an hypothesis to reconcile the algebra with the experimental results is to rewrite the equation for t as:

$$t(v) = \frac{2d(v,A)\sqrt{c^2 - v^2\sin^2(A)}}{(c^2 - v^2)} \qquad *EQ1$$

or changing the subject

$$d(v,A) = \frac{t(v)\ (c^2 - v^2)}{2\sqrt{c^2 - v^2\sin^2(A)}} \qquad *EQ2$$

NOTE: Henceforth assume that the general concepts of time t and distance d depending on v or A are true for any events and not just the elements of my apparatus (as Poincare did).

Consider events with the same velocity v and orientations of A = 90 degrees and A = 0 degrees in *EQ1*.

$$t(v) = \frac{2d(v,90)\sqrt{(c^2 - v^2)}}{(c^2 - v^2)} \quad \text{and} \quad t(v) = \frac{2d(v,0)c}{(c^2 - v^2)}$$

equating the two expressions and rearranging:

$$d(v,90)\,\frac{\sqrt{(c^2 - v^2)}}{c} = d(v,0)$$

$$\text{Let } L(v) = \frac{\sqrt{(c^2 - v^2)}}{c}$$

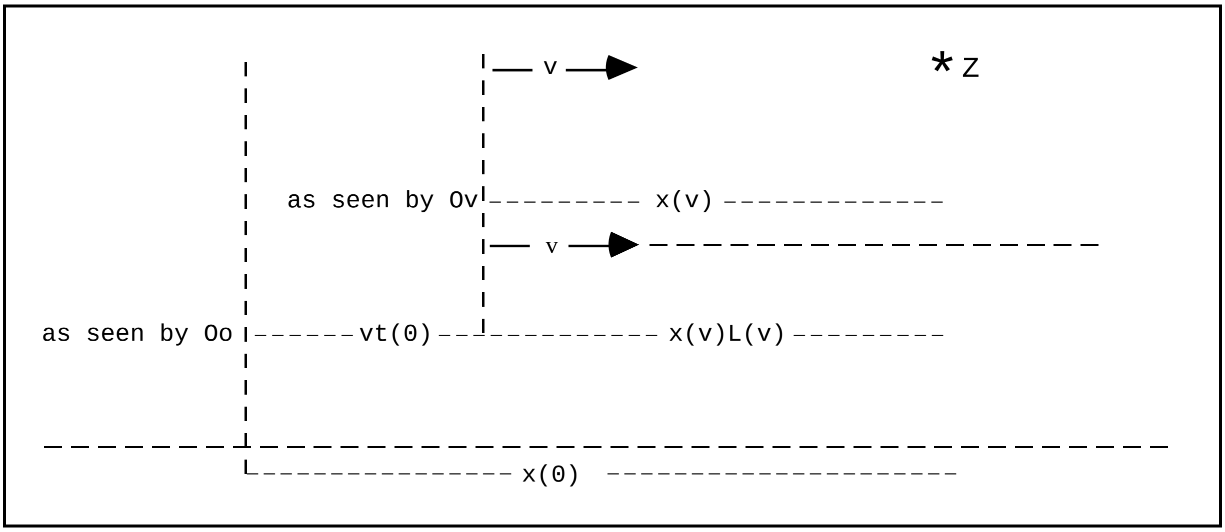
This factor will appear often, it is called the "Lorentz contraction factor". Note that L(v) < 1 when v < c.

Hence d(v,90).L(v) = d(v,0) *EQ3*

Which means that an observer should see the apparatus shortened in the direction of motion compared to its vertical dimension.

Since L(0) = 1, d(0,90) = d(0,0) *EQ4*

Which means that orientation has no effect on the distance measurement when the apparatus is "stationary" with respect to the observer.



Suppose that d(v,90) is constant for all values of v since the object being measured has a velocity of 0 at 90 degrees to the direction of motion, and is therefore "stationary" in this aspect.

in particular d(v,90) = d(0,90) *EQ5*

and also d(v,90) = d(0,0) (using *EQ4*)

Considered with *EQ3*

d(0,0).L(v) = d(v,0) *EQ6*

For v < c the distance measure of an object in the direction of motion is less as measured by a "stationary" observer than the measure made by an observer travelling with the object, or as measured afterwards when the object is "stationary" relative to an observer.

Using *EQ2*:

$$d(v,90) = \frac{t(v)\ (c^2 - v^2)}{2\sqrt{c^2 - v^2}} \quad \text{and} \quad d(0,90) = \frac{t(0)c^2}{2c}$$

$$d(v,90) = \frac{t(v)\sqrt{(c^2 - v^2)}}{2} \quad \text{and} \quad d(0,90) = \frac{t(0)c}{2}$$

since the left hand sides are equal (using *EQ5*)

t(0) = L(v).t(v) *EQ7*

For $v < c$ the time taken for an event occurring in a moving apparatus is greater as measured by a “stationary” observer than the measure made by an observer travelling with the apparatus, or as measured afterwards when the apparatus is “stationary” relative to an observer. This “time dilation effect” is supported by observations that some short-lived particles which result from cosmic collisions in the upper atmosphere live much longer than can be accounted for without supposing that their time rate is slower than ours. It is rather intriguing to realise that our time rate also looks slower to them.

Einstein proposed “The twins paradox” to popularise this effect, but let’s use triplets. Two of the triple go on identical journeys in fast spacecraft in opposite directions, eventually returning to the homebound member, each observes the other pair’s time to be going slower than their own by virtue of their high relative velocities. Upon returning home the two travellers are equally aged but the homebound one is older. The pair who journeyed, experienced identical accelerations and would have noticed that the account of ages was finalised while decelerating to home. Why is acceleration so kind to those who partake of it?

PART 2 – Relative Velocities

Suppose there are two observers, one “stationary”, named Oo the other moving at v in the “x” direction relative to Oo, named Ov. By previous arrangement, when Oo and Ov are at the same position, a light pulse leaves a point Z. Oo and Ov each receive the pulse after times of t(0) and t(v) respectively, both at an approach speed of c in accordance to principle 1. Oo and Ov calculate the co-ordinates of Z relative to themselves at these times as (x(0),y(0)) and (x(v),y(v)) respectively.

Hence $x(0) = ct(0)$ and $x(v) = ct(v)$

or $t(0) = x(0)/c$ and $t(v) = x(v)/c$ *EQ8*

In this time Oo sees Ov move a distance vt(0) and Ov is now x(v) units from Z which is seen by Oo as x(v).L(v) by *EQ6*

$$x(0) = vt(0) + x(v)L(v) \quad \text{or} \quad x(v) = \frac{x(0) - vt(0)}{L(v)} \qquad \text{*EQ9}$$

The observers now have formulae for calculating the position from z from each other’s perspective.

Also since there is no motion in the y direction y(0) = y(v), substituting *EQ8* into *EQ9*

$$ct(v) = \frac{ct(0) - vx(0)/c}{L(v)}$$

which becomes

$$t(v) = \frac{t(0) - vx(0)/c^2}{L(v)}$$

Next consider that the point z is moving with velocity Ux(0) and Ux(v) in the x direction, and Uy(0) and Uy(v) in the y direction, relative to Oo and Ov respectively.

that is $\frac{dx(0)}{dt(0)} = Ux(0)$ and $\frac{dx(v)}{dt(v)} = Ux(v)$

How are the velocities of z related? In Newtonian Physics

$$Ux(v) = Ux(0) - v$$

but now things are different.

$$Ux(v) = \frac{dx(v)}{dt(v)} = \frac{dx(v).dt(0)}{dt(0)\ dt(v)}$$

**EQ10*

$$Uy(v) = \frac{dy(v)}{dt(v)} = \frac{dy(v).dt(0)}{dt(0)\ dt(v)}$$

From **EQ9*

$$\frac{dx(v)}{dt(0)} = \frac{Ux(0) - v}{L(v)}$$

$$\frac{dt(v)}{dt(0)} = \frac{1 - vUx(0)/c^2}{L(v)} \quad \text{the inverse of} \quad \frac{dt(0)}{dt(v)}$$

$$\frac{dy(v)}{dt(0)} = \frac{dy(0)}{dt(0)} = Uy(0)$$

NOTE: In the y direction Ux(0) = dx(0)/dt(0) = 0. Also, since v is constant, L(v) is a constant with respect to time.

Hence **EQ10* becomes

$$Ux(v) = \frac{Ux(0) - v}{1 - Ux(0)v/c^2} \quad \text{and} \quad Uy(v) = Uy(0).L(v)$$ **EQ11*

This result indicates that objects externally observed as approaching at a combined speed in excess of c will appear to each other to be approaching at less than c. In particular if the approach speed Ux(0) = -c, then Ux(v) also is c, i.e. principle 1.

PART 3 – Mass and Energy

There are three assumptions that are retained from Newtonian Physics: mass, energy and momentum are conserved in a moving system just as they are in a "stationary" one. Let's consider the momentum of a ball as it moves in the "Y" direction. The momentum can be assessed by measuring its mass beforehand then timing it as it moves between two points to determine velocity which we will suppose is negligible compared to "c". For an observer moving past in the "X" direction (at 90 degrees to "Y") the "Y" component of the momentum of the ball should appear the same as for the observer sitting still relative to the measuring points. Assuming that both observers determine identical "Y" momentum for the ball and considering that the moving observer sees the ball move slower due to time dilation and yet is expected to calculate that its momentum is unaffected! (see *EQ11*) This is apparently another inconsistency. Suppose an object has mass m and y velocity of Uy(v) (which is also equal to Uy(0).L(v) by **EQ11*), its momentum in the Y direction is the same for both observers.

i.e. $m.Uy(v) = m.Uy(0)$ but $Uy(v) = Uy(0).L(v)$

Hence to "fix" the contradiction, we follow the precedent for time and suppose m depends on v. That is let:

$$m(v) = m(0)/L(v)$$ **EQ12*

For $v < c$ the mass of an object is greater as measured by a "stationary" observer than the measure made by an observer travelling with the object, or as measured afterwards when the object is "stationary" relative to an observer. Note that as speed builds towards c, the object's mass increases without bound, hence no object with non zero mass can be given sufficient energy to reach the speed of light.

Using *EQ12*

$$m(v)^2 = \left(\frac{m(0)^2}{L(v)}\right) = \frac{m(0)^2c^2}{c^2 - v^2} \qquad \text{substituting for } L(v)$$

Which is modified to

$$m(v)^2c^2 = m(0)^2c^2 + m(v)^2v^2$$

where $m(v).v$ is the momentum, commonly called "P"

$$m(v)^2c^2 = m(0)^2c^2 + p^2 \qquad \text{*EQ13}$$

Expanding $1/L(v)$ by Taylor series, *EQ12* becomes

$$m(v) = m(0)\ [1 + v^2/2c^2 + 3v^4/8c^4 + \ldots]$$

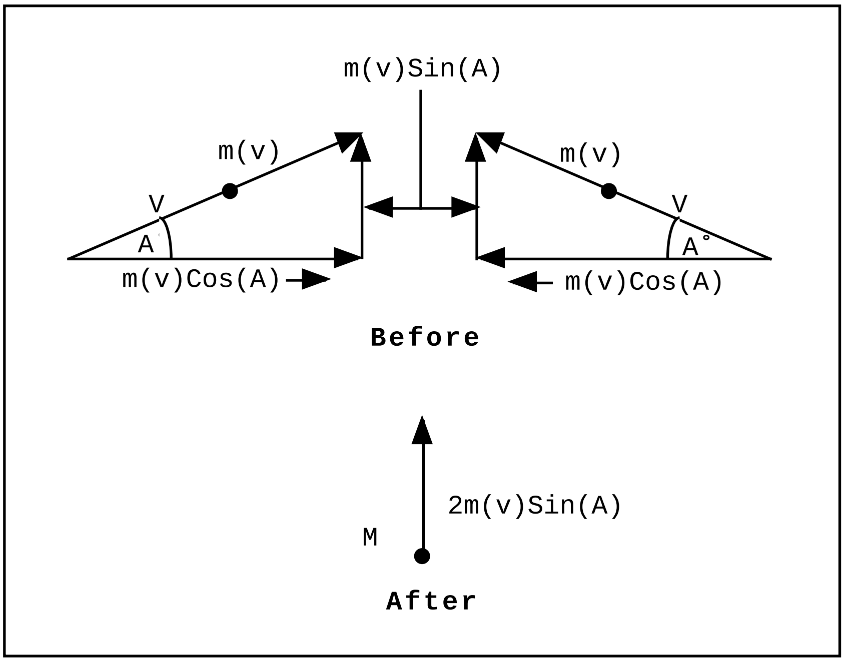
multiplying by c^2 and expanding

$$m(v)c^2 = m(0)c^2 + m(0)v^2/2 + 3m(0)v^4/8c^4 + \ldots \qquad \text{*EQ14}$$

The units of both sides of *EQ14* are those of energy and the familiar Newtonian Kinetic energy term appears second on the right hand side of the equation. The terms involving "v" can be equated to $p^2/m(v)$ using *EQ13*. Einstein proposed that *EQ14* was in fact an energy equation and that the total energy of a particle is $m(v)c^2$.

$$E = m(v)c^2 \qquad \text{*EQ15}$$

To suppose that the intrinsic energy of a mass is $m(0)c^2$ requires that an object of zero rest mass and velocity of less than c (since $1/L(c)$ is undefined) must have zero energy. Perhaps more justification for this conclusion is required. Consider two objects approaching each other at the same speed at an angle of A degrees to the X direction and colliding inelastically, that is they coalesce into one. The total momentum before the collision is $2m(v).v\sin(A)$ in the Y direction, hence this is the total momentum afterwards as well.



If the angle A is small then the resultant velocity is small and so the relativistic mass increase is negligible, the mass being close to 2m(0). However this is not the case if it is accepted that mass is conserved in the collision. The resultant mass is M=2m(v), if v was large then this is noticeably greater than 2m(0), the combined rest mass. Hence the mass of the resultant object depends on the speed or kinetic energy of its precursors. Moreover, should a particle of mass M fly apart into two fast parts, as in nuclear fission, each will dissipate its kinetic energy, hence velocity, hence relativistic mass as it slows through interactions with neighbouring particles. The energy released (KE) is a function of the mass deficit of the initial particle and its now stationary products. Taking a clue from *EQ15* this function is simply the multiplicative constant c^2.

Change of $KE = (m(v) - m(0))c^2$

It would seem that energy can be stored as mass and so this is evidence to support the claim that the intrinsic energy of a mass could be $m(0)c^2$. There are other implications of mass depending on creation velocity. The insides of a particle consist of the intrinsic masses of Protons etc. minus energy used to stick them together plus energy left over from the velocity of the creation of the particle, so what does a “masson” look like? Or is it that the insides of particles may emit protons etc. but that they do not actually exist inside the particle. As Richard Feynman put it, one can emit the word “cat” without having it ready made inside you, there is no finite supply of this word, only the energy used to make it. So it is for the “contents” of particles, the emission products do not necessarily exist internally.

A more useful and thought provoking version of *EQ15* is obtained by squaring and dividing by c^2 …

$$\frac{E^2}{c^2} = m(v)^2c^2$$

then using **EQ13*

$$\frac{E^2}{c^2} = m(0)^2c^2 + p^2$$

or

$$\frac{E^2}{c^2} - p^2 = m(0)^2c^2 \qquad \text{*EQ16}$$

In the preceding development, momentum was defined as a non zero mass multiplied by velocity. Although this is valid it does not exclude other forms of momentum. Consider an object with zero rest mass but with velocity c, calculating m(c) we have an ambivalent situation, both L(c) and m(0) are zero. Is their quotient zero, finite or infinite? If the answer is "finite" then there may be objects which have momentum and energy but only at the speed of light. Photons seem to fit this description. For example consider the momentum transferred from one particle to another by light, the source particle recoils on emission thus losing momentum, there is a slight travel time for the photon between particles, then the target particle gains momentum when hit by the photon. So if momentum is always conserved the photon must carry it. Hence let **EQ16* describe these as well.

PART 4 – Space-time

Redefine our units of measurement so that the speed of light is 1 unit (c = 1), this will simplify **EQ14* and help clarify the following:

$$E^2 - p^2 = m(0)^2 \qquad \text{*EQ17}$$

Energy minus momentum has a constant value (which we call rest mass).

Reviewing **EQ9* with c = 1 and hence $L(v) = \sqrt{1 - v^2}$

$$x(v) = \frac{x(0) - t(0)}{\sqrt{1 - v^2}}$$

and

$$t(v) = \frac{t(0) - vx(0)}{\sqrt{1 - v^2}}$$

Emulating **EQ17*

$$t(v)^2 - x(v)^2 = \frac{(t(0) - vx(0))^2 - (x(0) - vt(0))^2}{1 - v^2}$$

$$= \frac{t(0)(1 - v^2) - (1 - v^2)x(0)}{1 - v^2}$$

$$= t(0)^2 - x(0)^2$$

hence $t(v)^2 - x(v)^2$ = a constant

Omitting reference to v for simplicity

$$t^2 - x^2 = \text{a constant (independent of } v) \qquad \text{*EQ18}$$

$$E^2 - p^2 = \text{a constant (independent of } v) \qquad \text{*EQ19}$$

Note that the motion in general has x, y and z components, so x2 can be replaced by $x^2 + y^2 + z^2$ in **EQ18* and **EQ19*.

These equations have an intriguing interpretation. Recall that on a number plane the square of the distance between two points is:

$$d^2 = x^2 + y^2$$

In three dimensions the square of the distance is:

$$d^2 = x^2 + y^2 + z^2$$

where x, y, z denote the differences between the co-ordinates of the points. In Newtonian physics the value of d is the same no matter what the velocity of the source.

The theory of relativity, which apparently better reflects reality, predicts that such measures are not invariant but that the following is:

$$r^2 = t^2 - x^2 - y^2 - z^2$$

This formula is expressed using four parameters or dimensions, the three Newtonian ones (x, y, z), called the spatial dimensions, plus time. Note that the time dimension has a special place in the equation and cannot be interchanged (like x, y and z can) without effecting r. The four dimensional universe described by the four parameters is therefore called Space-time.

from **EQ17* $k^2 - E^2 - Px^2 - Py^2 - Pz^2$

where k is some constant, is also independent of changes due to velocity in the same way as $Px + Py + Pz$ for conservation of momentum in Newtonian Physics.

Note that these constants can be either positive or negative, unlike the Newtonian measure which is always positive.

If $t^2 > x^2 + y^2 + z^2$ then the measure is said to be time like, and r is a real number, otherwise, the measure is space like and r is an imaginary number. All events which can effect an object at a particular time have a space like measure relative to the object at that time.

Hence **EQ17* is the law of conservative of momentum, improved to meet the theoretical and experimental results. The implication is that Energy is a special kind of momentum in the time or fourth dimension. As an object increases in velocity its space and time characteristics blend together as seen by a “stationary” observer.

Throughout this essay assumptions have been made, where others might just as well have been. The justification for the accuracy of a theory does not lie in the elegance of the mathematics, but in confirmation by experiment. So far relativity has proved a better approximation to reality than any other theory. If a start is made with the concepts of conservation of momentum, mass and energy and distance in four dimensions, the variations in time, distance in three dimensions and mass are direct consequences.

PART 5 – The age of the Universe (relatively speaking)

Let’s have a look at a “for instance” arising from relativity. The universe is thought to have originated some 10 to 20 billion of our years ago in a big bang. This theory has been modified to include an “inflation” stage and is more recently subject to challenge. Anyway, the components of “the bang” have been flying apart ever since, gradually slowed by gravitational attraction. Due to the finite speed of light we observe objects at a distance from us as they were in the past when they were receding faster. Consequently, on a galactic scale, the further away we look the faster the recession velocity we observe, the “edge”, called the Hubble Limit, occurs where this velocity reaches the speed of light. We presume these observations to be the same for any observer anywhere and, despite the 3 dimensional geometric discomfort, that the edge is the big bang itself.

One of the more esoteric features of such a proposal is the initial state of the universe, theories about the nature of things down to minute fractions of a second after the big bang have developed to account for and predict observations. The question of what was about before or at time zero is often considered outside the realm of physics since “less than nothing” is a less than fulfilling answer. Perhaps Relativity can help to shed some light on the question or at least avoid it altogether.

Time actually moves slower in receding galaxies due to the time dilation effect, in every aspect of its effect on us this slower rate is significant, it is the “real” time from our viewpoint. If we observe a galaxy receding at `0.8c` then the actual passage of time as measured by us will be slowed by a factor of `0.6`. Hence if we observe an event in our galaxy that takes `9` years, the same event would be seen by us to take `9/0.6 = 15` years in the receding galaxy.

This actual slowing of time and consequent extended age increases without bound as the recession speed approaches the speed of light. The open ended scale often used to represent the evolution of the universe so as to give similar space to the first busy fractions of a second and subsequent billions of years may therefore be a more representative model of the universe from the viewpoint of any observer, than by encompassing all with one’s own parochial time zone. Using our common sense time on such a scale is conceptually like trying to use a metre ruler to measure the circumference of the Earth, it is “straight” and so is the Earth on a local scale, but the Earth is “bent” on a large scale and so the ruler is inappropriate for the global scale. The conclusion I draw from the time dilation effect on a universal scale is that time is the wrong tool to determine a limit for the universe. In the real measure of time, which is subject to dilation with velocity and therefore distance, the universe is of unbounded age.